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Synchrotron Motion with Radiation Reaction

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Abstract

It is shown that the longitudinal velocity of a charged particle moving in a uniform magnetic field, and obeying Dirac–Lorentz relativistic equation of motion with radiation reaction is constant. Suitable approximate methods, which give fairly accurate results, are used to obtain the expression for velocity and displacement along the transverse section. They describe the motion completely up to a correcting factor

$$1 + 0\left\{\left(\frac{e^3 B}{m^2 c^4}\right)^2\right\}; \qquad \frac{e^3 B}{m^2 c^4} \simeq 10^{-16} B$$

for electrons, B in G.

1. *Introduction*

The problem of the motion of charge particles moving in an intense magnetic field has been studied for quite a long time because of its application in accelerators. Recently, there has been a revival of interest in the problem due to its application in astrophysics. Since the radiation which is emitted is quite intense and the field strength is also very high, it is imperative to include the effect of radiation reaction on the motion of the particles. Further, the energy of the particles are also very high so that usual non-relativistic approximations are no longer tenable. Hence, one is obliged to integrate the Dirac–Lorentz relativistic equation of motion with radiation reaction, which is given by

$$\dot{\mathbf{v}} - \epsilon(\ddot{\mathbf{v}} + \mathbf{v}\zeta) = \frac{e}{mc}\mathbf{v} \times \mathbf{B} \tag{1.1}$$

$$\dot{l} - \epsilon(\ddot{l} + l\zeta) = 0 \tag{1.2}$$

Dots denote differentiation with respect to proper time, τ, of the particle and

$$\left.\begin{aligned} c\mathbf{v} &= \dot{\mathbf{r}}, \qquad l = (1 + \mathbf{v}.\mathbf{v})^{1/2} \\ \zeta &= \dot{\mathbf{v}}.\dot{\mathbf{v}} - \frac{(\mathbf{v}.\dot{\mathbf{v}})^2}{1 + \mathbf{v}.\mathbf{v}}, \qquad \epsilon = \frac{2e^2}{3mc^3} \end{aligned}\right\} \tag{1.3}$$

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In order to solve the equation of motion, we proceed exactly in the same manner as that of the motion in a constant electric field (Sen Gupta, 1971). Thus we are interested in those solutions which are regular, as $\epsilon \to 0$. In a short communication, the author (Sen Gupta, 1970) has shown that the longitudinal velocity $cv_\parallel$ of the particle is constant. This is because of the fact that

$$\exp(-\tau/\epsilon)(\epsilon\mathbf{k}.\dot{\mathbf{v}} - \mathbf{k}.\mathbf{v}) = \text{constant} \tag{1.4}$$

is an exact integral and the constant is necessarily zero, thus

$$\mathbf{k}.\frac{d\mathbf{r}}{dt} = \text{constant} = cv_\parallel \tag{1.5}$$

In the next section we develop a suitable approximate method to integrate the equation of motion along the transverse plane. Section 3 is devoted to the discussion of the locus. It can be emphasised that the method is not the usual perturbation method, which is not applicable in this highly non-linear problem. After some expedient remarks about the nature of radiation emitted in Section 4, the paper concludes with a discussion.

2. *The Motion Along the Transverse Section*

Equation (1.1) can be expressed as

$$\dot{\mathbf{v}} - \epsilon(\ddot{\mathbf{v}} + \mathbf{v}\zeta) = \Omega\mathbf{v} \times \mathbf{k} \tag{2.1}$$

where $\Omega = eB/mc$ is the Larmor frequency. It does not admit of any simple integral other than (1.5) and it will be extremely difficult to find the exact solution. We will try to find the integrals with successive degree of approximation. For this purpose we first express equation (2.1) in terms of quantities which are of direct physical importance, namely $E = \sqrt{(1 + \mathbf{v}.\mathbf{v})}$ (the particle energy in unit of its rest energy) and ζ. Thus

$$\dot{E} - \epsilon(\ddot{E} - E\zeta) = 0 \tag{2.2}$$

$$\zeta - \tfrac{3}{2}\epsilon\dot{\zeta} + \frac{\epsilon^2}{2}\ddot{\zeta} = \Omega^2\{(1 - v_\parallel^2)E^2 - 1\} \tag{2.3}$$

Next, we observe that the characteristic time concomitant to the radiation reaction $\epsilon \simeq 10^{-23}$ sec but the Larmor frequency, even with $B \simeq 10^9$ G for electrons, $\Omega \simeq 10^{15}$ sec^{-1}, so that $\epsilon\Omega \simeq 4 \times 10^{-8}$. Thus, it is quite meaningful to consider $\epsilon\Omega$ as small and seek solutions whose terms are successively of higher orders in $\epsilon\Omega$.

Since E is constant when $\epsilon = 0$, $\dot{E}$ is at least first-order in ϵ and the second term in equation (2.2) is at least of second-order in ϵ. Thus, for first-order approximation one can write in the third term of equation (2.2),

$$\zeta = \Omega\{(1 - v_\parallel^2)E^2 - 1\} \tag{2.4}$$

to obtain

$$\dot{E} + \epsilon\Omega E\{(1 - v_\parallel^2)E^2 - 1\} = 0 \tag{2.5}$$

So that

$$\frac{1}{E^2} = 1 - v_\parallel^2 - v_\perp^2 \exp(-2\epsilon\Omega\tau) \tag{2.6}$$

where $cv_\perp$ is the projection of initial velocity on the transverse plane. Let its direction be $\mathbf{j}$. By integrating $dt/d\tau = E$, τ can be expressed in terms of t,

$$\cosh(\epsilon\Omega^2\sqrt{(1 - v_\parallel^2)}\,t + \phi) = \exp(\epsilon\Omega^2\tau)\cosh\phi \tag{2.7}$$

where

$$\cosh\phi = v_\perp^{-1}\sqrt{(1 - v_\parallel^2)} \tag{2.8}$$

The constant of integration is so chosen that $\tau = 0$ at $t = 0$. From equation (2.6)

$$\mathbf{k} \times \mathbf{v}.\mathbf{k} \times \mathbf{v} = E^2 v_\perp^2 \exp(-2\epsilon\Omega^2\tau) \tag{2.9}$$

In order to obtain a better approximation we add to the right-hand side of equations (2.4) and (2.5), $\frac{3}{4}\epsilon\dot{\zeta}$ and $\epsilon\ddot{E}$ respectively. The expressions for which are obtained from equations (2.4) and (2.5) by differentiation. On integration one obtains

$$\left\{\frac{(1 - v_\parallel^2)E^2 - 1}{(1 - v_\parallel^2)E_0^2 - 1}\right\}^{1+5\epsilon^2\Omega^2} \left(\frac{E_0}{E}\right)^{2(1-\epsilon^2\Omega^2)} = \exp(-2\epsilon\Omega^2\tau) \tag{2.10}$$

Hence,

$$E^{-2} = (1 - v_\parallel^2) - v_\perp^2 \exp(-2\epsilon\Omega^2\tau) f(\tau)^{-4\epsilon^2\Omega^2} \tag{2.11}$$

where

$$f(\tau) = E_0^2[1 - v_\parallel^2 - v_\perp^2 \exp(-2\epsilon\Omega^2\tau)] \tag{2.12}$$

($E_0 = E(0)$). This introduces in the equations (2.6), (2.7) and (2.9) a correcting factor $1 + O(\epsilon^2\Omega^2)$. Since the upper bound of the coefficient of $\epsilon^2\Omega^2$ is less than unity the expressions (2.6)–(2.9) are valid to a good degree of accuracy. It may be noted that even with $B \simeq 4 \times 10^{13}$ G which is the threshold field for quantum effect, $\epsilon^2\Omega^2 \simeq 10^{-8}$. Further, it is easy to see that each step of the successive approximation introduces a factor $1 + O(\epsilon^2\Omega^2)$. Thus, equation (2.6) for E^2 and equation (2.9) for ($\mathbf{k} \times \mathbf{v}.\mathbf{k} \times \mathbf{v}$) give respectively their dependence on time fairly accurately and they are valid for all time. As it is expected E asymptotically tends to $(1 - v_\parallel^2)^{-1/2}$ and $|\mathbf{k} \times \mathbf{v}|$ to 0, so that the radiation emitted by the particle is totally at the expense of its energy due to transverse component of initial velocity.

3. *The Locus*

From equation (1.1) one obtains

$$\dot{\mathbf{v}}.\mathbf{v} \times \mathbf{k} - \epsilon\ddot{\mathbf{v}}.\mathbf{v} \times \mathbf{k} = \Omega\mathbf{v} \times \mathbf{k}.\mathbf{v} \times \mathbf{k} \tag{3.1}$$

Hence,

$$\frac{d}{d\tau}\tan^{-1}\frac{\mathbf{j}\times\mathbf{k}.\mathbf{v}}{\mathbf{j}.\mathbf{v}} = \Omega\{1 + 0(\epsilon^2\Omega^2)\} \tag{3.2}$$

and the transverse component of velocity, $c\mathbf{v}_\perp = \dot{\mathbf{r}}_\perp$

$$\mathbf{v}_\perp = v_\perp \exp(-\epsilon\Omega^2\tau) E(\mathbf{j}\cos\Omega\tau + \mathbf{j}\times\mathbf{k}\sin\Omega\tau) \tag{3.3}$$

on further integration this leads to

$$\mathbf{r} = \mathbf{k}cv_\parallel t + cv_\perp E\exp(-\epsilon\Omega^2\tau)\{\mathbf{j}\sin\Omega\tau - \mathbf{j}\times\mathbf{k}\cos\Omega\tau - \epsilon\Omega E^2(1 - v_\parallel^2)\times$$
$$\times(\mathbf{j}\cos\Omega\tau + \mathbf{j}\times\mathbf{k}\sin\Omega\tau)\} + \mathbf{j}\epsilon c E_0^3 v_\perp(1 - v_\parallel^2) \tag{3.4}$$

The constants of integration are so chosen that at $t = 0$, $\mathbf{r} = \mathbf{j}\times\mathbf{k}cv_\perp E_0\Omega^{-1}$ so that $\mathbf{r} = 0$ lies on the axis of the spiral orbit for $\epsilon = 0$. The transverse part of the displacement is expressed as a function the proper time τ. In order to express this in terms of t, we first observe that the factor $\exp(2\epsilon\Omega^2\tau)$ can be directly expressed in terms of t from equation (2.7), though, a formal expression for $\cos\Omega\tau$ or $\sin\Omega\tau$ can be written but it is very involved. However, one can write for $t \ll \epsilon^{-1}\Omega^{-2}$

$$\exp(i\Omega\tau) = \exp i\Omega\tau\{E_0^{-1} + \epsilon\Omega^2 v_\perp^2 t + 0(\epsilon^2\Omega^2)\} \tag{3.5}$$

Basically the motion in the transverse plane is oscillatory. The amplitude is gradually decreasing. The time interval, T, between two successive t for which $\exp(i\Omega\tau)$ is the same, is

$$T_n = 2\pi E_0\Omega^{-1}\{1 - 4\pi^2\epsilon\Omega(2n+1)v_\perp^2 E_0^2 + 0(\epsilon^2\Omega^2)\} \tag{3.6}$$

where n is a positive integer, thus for small n, T is independent of 't' and the expression (3.5) may be taken as periodic with period $2\pi E_0\Omega^{-1}$ so long as the time interval is very much smaller than $\epsilon^{-1}\Omega^{-2} \simeq 10^{-3}$ sec for $B = 10^6$ G. The expression (3.4) for $\mathbf{r}$ shows that in the transverse section the particle asymptotically reaches to

$$\mathbf{r}(\alpha) = \mathbf{j}\epsilon E_0^3 cv_\perp(1 - v_\parallel^2) \tag{3.7}$$

Thus the particle does not reach the axis of the spiral orbit for $\epsilon = 0$ but this deviation is only along $\mathbf{j}$ and depends on the magnitudes $v_\parallel$ and $v_\perp$; further, it is independent of B.

4. *The Synchroton Radiation*

The nature of the radiation emitted by the particle at great distance from the particle is predominantly dependent on the acceleration. It is obtained from expression (3.4), as

$$\frac{d^2\mathbf{r}}{dt^2} = \frac{cv_\perp\Omega}{2E}\exp[\Omega\tau(i - \epsilon\Omega)](1 + i\epsilon\Omega)(\mathbf{j}\times\mathbf{k} + i\mathbf{j}) + c.c \tag{4.1}$$

The nature of the variation of the oscillatory parts with time namely $\exp(\pm i\Omega\tau)$ has already been discussed in the previous section. From equation (2.7), one observes that the variation of the remaining factor, which slowly and continuously decreases to zero, is very small in the interval $2\pi E_0 \Omega^{-1}$, hence, the radiation emitted gradually decreases with time. The angular distribution of such radiation has been worked out in detail by Schwinger (1949). Further, the expression (4.1) shows that the radiation reaction introduces a very small phase difference between the two opposite circularly polarised radiations.

5. *Discussion*

Our investigation shows that the motion asymptotically tends to uniform velocity motion along the lines of force. This can be directly inferred from equation (1.1) which admits of a steady state solution, namely $\mathbf{v} \times \mathbf{k} = 0$. The total displacement along the transverse section is

$$cv_{\perp} E_0\{\mathbf{j}\epsilon(1 - v_{\parallel}^2)E_0^2 - \mathbf{j} \times \mathbf{k}\Omega^{-1}\}$$

The method of successive approximation used in the paper is quite different from the usual perturbation expansion in terms of ϵ as can be seen from the expressions (2.6), (2.9) and (2.12). Such expansions will be valid only for an extremely small interval of time. On the other hand our expressions are valid for all time and are accurate up to a factor $1 + O(\epsilon^2\Omega^2)$. The error involved is practically negligible for almost all physical field intensities. Hence, so long as quantum effects are not introduced our results are accurate to a sufficiently high degree. The only expression which is of restricted validity is (3.5); this is only because we have truncated an expansion at the second term. If we want the approximate period in the region of the instant t_0, we should get them from the difference of two successive roots of equation (2.7),

$$\cosh(\epsilon\Omega^2\sqrt{(1 - v_{\parallel}^2)}\,t_n + \phi) = \exp(2\pi n\epsilon\Omega)\cosh\phi \qquad (5.1)$$

for n and $n+1$ such that $t_n < t_0 < t_{n+1}$. This shows that approximate periods tend to the constant

$$T(t_0 \rightarrow \alpha) = 2\pi E_\alpha \Omega^{-1} \qquad (5.2)$$

where E_α is

$$E(t \rightarrow \alpha) = (1 - v_{\parallel})^{-1/2} \qquad (5.3)$$

Further, our approximation is quite distinct from the usual non-relativistic approximation in which one puts $\zeta = 0$. But we have started with ζ given by equation (2.4). It may not be irrelevant to mention that Shen (1970) has also started from an approximate expression for ζ, but in order to obtain the approximate expression for ζ, he has assumed, without justification, the transformation property of the field quantities and the equation of motion, in non-inertial frames of reference, namely the rest frame of the particle. This is one of the reasons that he finds the longitudinal velocity is also decreasing with time. Equation (1.5), is an exact integral and if one takes the constant of integration to be non-zero the magnitude of the longitudinal velocity would increase indefinitely with time instead of decreasing with time.

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